

A remark on representations of infinite symmetric groups

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We simplify construction of Thoma representations of an infinite symmetric group.

1. Spherical representations. Let G be a group, K a subgroup. The pair (G, K) is called *spherical* if for any irreducible unitary representation ρ of G the dimension of the subspace of K -fixed vectors is ≤ 1 . A unit vector of this subspace is called a *spherical vector*. Recall that the *spherical function* of an irreducible spherical representation ρ of G is given by

$$\Phi(g) := \langle \rho(g)v, v \rangle,$$

where v is the unit K -fixed vector.

2. The double of symmetric group. Let Ω be a countable set. A permutation of Ω is called *finite* if it fixes all but finite number elements of Ω . Denote by $S(\Omega)$ the group of finite permutations of a countable set Ω . We also use an alternative notation S_∞ for such groups.

Now let $G = S_\infty \times S_\infty$ be the product of two copies of S_∞ , let $K \simeq S_\infty$ be the diagonal subgroup. By [5], the pair (G, K) is spherical.

3. Thoma formula. Let G, K be the same as in the previous subsection. Representations of G spherical with respect to K are parametrized by collection of positive numbers

$$\alpha_1 \geq \alpha_2 \geq \dots, \quad \beta_1 \geq \beta_2 \geq \dots, \quad \sum \alpha_i + \sum \beta_j \leq 1,$$

finite and empty collections are admissible. Spherical functions are given by the formula

$$(1) \quad \Phi_{\alpha, \beta}(\sigma, \tau) = \prod_{k=2}^{\infty} \left(\sum_i \alpha_i^k + (-1)^{k-1} \sum_j \beta_j^k \right)^{\text{number of cycles of } \sigma\tau^{-1} \text{ of length } k}$$

(the product is finite), Thoma (1964) formulated this statement on another language (see [6], [5]).

Explicit constructions of Thoma representations were obtained by Vershik and Kerov (see [7], another version was done by Vershik [8]). Olshanski [5] proposed a transparent construction (which also is a modification of [7]), but it does not cover all possible values of parameters, we add a missing element to his construction.

4. Olshanski construction. Let

$$(2) \quad \sum \alpha_i + \sum \beta_j = 1$$

Consider a Hilbert space $H = H_{\overline{0}} \oplus H_{\overline{1}}$. Fix an orthonormal basis e_i in $H_{\overline{0}}$ and f_j in $H_{\overline{1}}$. Consider a unit vector $\xi \in H \otimes H$ given by

$$\xi := \sum_i \alpha_i^{1/2} e_i \otimes e_i + \sum_j \beta_j^{1/2} f_j \otimes f_j.$$

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We consider the infinite super-tensor product²

$$(3) \quad \mathcal{H} := (H \otimes H, \xi) \otimes (H \otimes H, \xi) \otimes (H \otimes H, \xi) \otimes \dots$$

(recall that the definition of an infinite tensor product requires distinguished unit vectors). The subgroup $S_\infty \times e \subset G$ acts in \mathcal{H} by permutations of first factors in brackets $(H \otimes H, \xi)$, the group $e \times S_\infty$ acts by permutations of the second factors. The diagonal subgroup K acts by permutations of the whole brackets $(H \otimes H, \xi)$ and $\xi^{\otimes \infty}$ is the K -spherical vector.

However, the condition (2) is essential³, the same difficulty arises for other spherical pairs considered by Olshanski [5] and in a more general setting discussed in [3], [4].

5. Products of spherical functions. Consider two irreducible spherical representations of G corresponding to parameters α, β and α', β' . It can be easily shown that their tensor product contains only one K -fixed vector⁴, and the spherical function is the product of spherical functions, it has the form (1) with parameters

$$\tilde{\alpha} = \{\alpha_i \alpha'_k\}, \{\beta_j \beta'_l\}, \quad \tilde{\beta} = \{\alpha_i \beta'_l\}, \{\beta_j \alpha'_k\}.$$

Therefore to construct all spherical representations it suffices to construct a representation⁵ corresponding to a single-element collection α and empty collection β , the spherical function of this representation is

$$(4) \quad \Psi(\sigma, \tau) = \alpha^{\text{number of } i \text{ such that } \sigma i \neq \tau i}.$$

6. Group of affine isometries and Araki scheme. For details, see, e.g., [2], V.1.6-7, X.1. Denote by F_n the Hilbert space of holomorphic functions on \mathbb{C}^n with the inner product

$$\langle f, g \rangle = \frac{1}{\pi^n} \int_{\mathbb{C}^n} f(z) \overline{g(z)} e^{-\langle z, z \rangle} dz d\bar{z}.$$

Consider the natural embeddings $J_n : F_n \rightarrow F_{n+1}$ given by

$$J_n f(z_1, \dots, z_n, z_{n+1}) = f(z_1, \dots, z_n).$$

Evidently, J_n is an isometric embedding. We consider the union of the chain

$$\dots \longrightarrow F_n \longrightarrow F_{n+1} \longrightarrow \dots$$

and its completion \mathbf{F}_∞ (the *boson Fock space*), see, e.g., [2], VI.1.

Consider a real Hilbert space H . Denote by $O(H)$ the group of orthogonal (i.e., real unitary) operators. Consider the group $\text{Isom}(H)$ generated by $O(H)$ and translations, we get a semi-direct product $\text{Isom}(H) = O(H) \ltimes H$, this group acts in H by affine transformations

$$h \mapsto Ah + v, \quad \text{where } A \in O(H), v \in H.$$

²Consider a \mathbb{Z}_2 -graded space $V = V_{\overline{0}} \oplus V_{\overline{1}}$. Then $V \otimes V$ is the usual tensor product, but the operator of transposition of factors is another: for homogeneous elements v, w we have $v \otimes w \rightarrow (-1)^\varepsilon w \otimes v$, where $\varepsilon = 1$ if $v, w \in V_{\overline{1}}$ and $\varepsilon = 0$ if at least one of the vectors v, w is contained in $V_{\overline{0}}$. On n -factor product $V \otimes \dots \otimes V$ we have an action of the symmetric group S_n , any permutation is a product of transpositions, and action of a transposition was described above.

³Otherwise the length of ξ is not 1 and (3) is not well-defined.

⁴but this representation can be reducible

⁵Another construction of this representation can be found in [1].

Next, we construct the linear representation $Exp(\cdot)$ of $\text{Isom}(\ell_2)$ in the Fock space \mathbf{F}_∞ . Orthogonal transformations act in the Fock space by

$$Exp(A)f(z) = f(zA), \quad A \in O(\ell_2),$$

translations by

$$Exp(v)f(z) = f(z+v)e^{-\langle z,v \rangle - \frac{1}{2}\langle v,v \rangle}, \quad v \in \ell_2.$$

Thus we get an irreducible unitary representation of $\text{Isom}(\ell_2)$. The function $f(z) = 1$ is $O(\ell_2)$ -invariant, the spherical function is $e^{-\frac{1}{2}\langle h,h \rangle}$.

One of the most common ways⁶ (*Araki scheme*) to construct representations of infinite-dimensional groups G is embeddings of G to the group of isometries of Hilbert space and restrictions of the representation $Exp(\cdot)$ to G .

Let we have a unitary representation U of a group G in a Hilbert space H . Let $\Xi : G \rightarrow H$ be a function satisfying

$$(5) \quad \Xi(g_1g_2) = T(g_1)\Xi(g_2) + \Xi(g_1).$$

Then affine isometric transformations

$$(6) \quad \tilde{U}(g)h = U(g)h + \Xi(g)$$

satisfy

$$\tilde{U}(g_1)\tilde{U}(g_2) = \tilde{U}(g_1g_2)$$

and we get embedding $G \rightarrow \text{Isom}(H)$ (this is straightforward). For a fixed vector $\eta \in H$ the function

$$(7) \quad \Xi(g) := U(g)\eta - \eta$$

satisfies the equation (5). This solution of (5) is not interesting because such correction $\Xi(\cdot)$ is equivalent to a change of origin of coordinates.

Now let G acts by linear transformations of a larger linear space $\hat{H} \supset H$, $\eta \in \hat{H} \setminus H$. If $U(g)\eta - \eta \in H$ for all g , then we get a nontrivial affine isometric action of G .

7. The construction for the double of the symmetric group. Consider the space ℓ_2 with basis e_j . The group $G = S_\infty \times S_\infty$ acts in $\ell_2 \otimes \ell_2$ by linear transformations

$$U(\sigma, \tau)e_i \otimes e_j = e_{\sigma i} \otimes e_{\tau j}$$

We fix $s > 0$, set

$$\eta := s \sum_{j=1}^{\infty} e_j \otimes e_j$$

and define $\Xi(\sigma, \tau)$ by (7). Note that $\eta \notin \ell_2 \otimes \ell_2$ but $\Xi(\sigma, \tau) \in \ell_2 \otimes \ell_2$. Also, $\Xi(\sigma, \sigma) = 0$. We define affine isometric action of $G = S_\infty \times S_\infty$ by (6) and restrict the representation of $\text{Isom}(\ell_2 \otimes \ell_2)$ to G . Then the function $f = 1$ is a unique K -fixed vector, the spherical function is (4) with $\alpha = e^{-s^2}$.

8. Symmetric group and hyper-octahedral subgroup. Consider two copies of \mathbb{N} , say \mathbb{N}_+ and \mathbb{N}_- . Denote their points by $1_+, 2_+, \dots, 1_-, 2_-, \dots$. Let $G_1 = S(\mathbb{N}_+ \sqcup \mathbb{N}_-) =: S_{2\infty}$. The *hyperoctahedral group* K_1 is the subgroup in G_1 consisting of permutations such that for any $j \in \mathbb{N}$ the pair $(\sigma j_+, \sigma j_-)$ has the form (m_+, m_-) or (m_-, m_+) . Evidently, K_1 is a semidirect product, $K_1 = S_\infty \ltimes \mathbb{Z}_2^\infty$.

⁶See numerous examples in [2], Sections VIII.6.8, IX.1.6, IX.2.5, IX.5.4, IX.1.4, IX.1.5, IX.3.12, IX.4.6, F.4.

The pair (G_1, K_1) is spherical, see [5].

Now we consider the Hilbert spaces $\ell_2(\mathbb{N}_\pm)$ with bases e_j^\pm . Consider the Hilbert space

$$H = (\ell_2(\mathbb{N}_+) \oplus \ell_2(\mathbb{N}_-)) \otimes (\ell_2(\mathbb{N}_+) \oplus \ell_2(\mathbb{N}_-))$$

The group $G_1 = S(\mathbb{N}_+ \sqcup \mathbb{N}_-)$ acts in this space in a natural way (on each tensor factor). Next, we define the vector

$$\eta := s \cdot \sum_j (e_j^+ \oplus e_j^- + e_j^- \oplus e_j^+)$$

and construct an embedding of $G_1 = S(\mathbb{N}_+ \sqcup \mathbb{N}_-)$ to $\text{Isom}(H)$ as above.

This gives K_1 -spherical representations of G_1 , which were not covered by explicit construction of [5].

9. One more example. Consider the pair $G_2 \supset K_2$, where $G_2 = G_1$ and $K_2 \subset K_1$ is the group of permutations σ sending any ordered pair (j_+, j_-) to a pair (m_+, m_-) . In fact,

$$K_2 = S_\infty \subset S_\infty \ltimes \mathbb{Z}_2^\infty = K_1$$

This pair is spherical (see [4], note that this fact has not counterpart for finite symmetric groups). Now we fix real parameters s, t , set

$$\eta = s \sum e_j^+ \otimes e_j^- + t \sum e_j^- \otimes e_j^+$$

and repeat the same arguments.

10. Triple products. Now let $G_3 = S_\infty \times S_\infty \times S_\infty$, $K_3 \simeq S_\infty$ be the diagonal. This pair is spherical, see [3], [4]. We set

$$H := \ell_2 \otimes \ell_2 \otimes \ell_2$$

and

$$\eta = s \sum_j e_j \otimes e_j \otimes e_j.$$

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